Computing the moment polynomials of the zeta function

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Abstract

We describe a method to accelerate the numerical computation of the coefficients of the polynomials $P_k(x)$ that appear in the conjectured asymptotics of the $2k$-th moment of the Riemann zeta function. We carried out our method to compute the moment polynomials for $k \le 13$, and used these to experimentally test conjectures for the moments up to height 10^8.

1 Introduction

For positive integer k, and any $\epsilon > 0$, Conrey, Farmer, Keating, Rubinstein, and Snaith conjectured [CFKRS] that

$$\int_0^T |\zeta(1/2+it)|^{2k} dt = \int_0^T P_k\left(\log(t/(2\pi))\right) dt + O\left(T^{1/2+\epsilon}\right), \tag{1}$$

with the constant in the O term depending on k and ϵ.

In the above equation P_k is the polynomial of degree k^2 given implicitly by the $2k$-fold residue

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$$P_k(x) = \frac{(-1)^k}{k!^2}\frac{1}{(2\pi i)^{2k}}\oint\cdots\oint\frac{G(z_1,\ldots,z_{2k})\Delta^2(z_1,\ldots,z_{2k})}{\prod_{i=1}^{2k} z_i^{2k}} \times e^{\frac{x}{2}\sum_{i=1}^{k} z_i - z_{i+k}}\, dz_1\ldots dz_{2k}, \qquad (2)$$

with the path of integration over small circles about $z_i = 0$, where

$$\Delta(z_1,\ldots,z_m) = \prod_{1\le i<j\le m}(z_j - z_i) = \left|z_i^{j-1}\right|_{m\times m}$$

denotes the Vandermonde determinant,

$$G(z_1,\ldots,z_{2k}) = A_k(z_1,\ldots,z_{2k})\prod_{i=1}^{k}\prod_{j=1}^{k}\zeta(1+z_i-z_{j+k}), \qquad (3)$$

and A_k is the Euler product

$$\begin{aligned}&A_k(z_1,\ldots,z_{2k})\\&=\prod_p\prod_{i,j=1}^{k}(1-p^{-1-z_i+z_{k+j}})\int_0^1\prod_{j=1}^{k}\left(1-\frac{e(\theta)}{p^{\frac{1}{2}+z_j}}\right)^{-1}\left(1-\frac{e(-\theta)}{p^{\frac{1}{2}-z_{k+j}}}\right)^{-1}d\theta. \qquad (4)\end{aligned}$$

Here $e(\theta) = \exp(2\pi i\theta)$.

We denote the coefficients of $P_k(x)$ by $c_r(k)$:

$$P_k(x) =: \sum_{r=0}^{k^2} c_r(k)x^{k^2-r}. \qquad (5)$$

In order to arrive at this conjecture, CFKRS considered a more general moment problem with 'shifts'. Because this general setting was central to our computation, we describe their conjecture with shifts below. Write the functional equation of zeta as

$$\zeta(s) = \chi(s)\zeta(1-s),$$

where

$$\chi(s) := \pi^{s-1/2}\Gamma((1-s)/2)/\Gamma(s/2).$$

It is a little more convenient to work with the Hardy Z-function, whose functional equation and approximate functional equation are expressed more symmetrically than that of the zeta function. It is defined as

$$Z(s) = \chi(s)^{-1/2}\zeta(s),$$

and satisfies

i) $Z(s) = Z(1-s)$ (because $\chi(s)\chi(1-s) = 1$),

ii) $Z(1/2+it) \in \mathbb{R}$ for $t \in \mathbb{R}$,

iii) $|Z(1/2+it)| = |\zeta(1/2+it)|$.

CFKRS took as their starting point the shifted moments:

$$M(\alpha_1,\ldots,\alpha_{2k}) := \int_0^T Z(1/2+it+\alpha_1)\cdots Z(1/2+it+\alpha_{2k})\,dt,$$

where $\alpha_j \in \mathbb{C}$ are distinct and satisfy $-1/4 < \Re\alpha_j$. When $\alpha = 0$ the integrand is $|\zeta(1/2+it)|^{2k}$.

Substituting the approximate functional equation into each factor of the above integrand

$$Z(s) = \chi(s)^{-1/2} \sum_{n\le\sqrt{\frac{t}{2\pi}}} \frac{1}{n^s} + \chi(1-s)^{-1/2} \sum_{n\le\sqrt{\frac{t}{2\pi}}} \frac{1}{n^{1-s}} + O(t^{-\sigma/2}),$$

$s = \sigma + it$, $0 < \sigma < 1$, CFKRS applied the following *heuristic* steps:

a) Ignore the $O(t^{-\sigma/2})$ and expand the product to get 2^{2k} terms, each a product $2k$ sums.

b) Of the 2^{2k} terms, only the terms with the same number of s's and $1-s$'s contribute to the asymptotics. Reasoning: $\chi(s)$ is highly oscillatory, so cancellation occurs unless each s gets paired with a $1-s$.

c) For any such term, only the diagonal ('$m_1m_2\ldots m_k = n_1n_2\ldots n_k$') contributes when the sums are multiplied out.

d) Extend the truncated diagonal sums to infinity, replacing the sums that diverge with their analytic continuation (the assumption we stated earlier, $-1/4 < \Re\alpha_j$, is used when obtaining the analytic continuation of the diagonal sums).

The steps in this heuristic recipe are not justifiable. In fact, the terms that are dropped cannot be neglected individually, and it appears that some sort of cancellation takes place amongst these terms so that, in the end, the above steps do apparently result in a correct conjecture.

Let

$$H(z_1,\dots,z_{2k};x) := \exp\left(\frac{x}{2}\sum_{1}^{k} z_j - z_{j+k}\right) G(z_1,\dots,z_{2k}). \tag{6}$$

The first conjectured asymptotic formula of CFKRS, which we refer to as the combinatorial sum, for shifted moments reads:

$$M(\alpha_1,\dots,\alpha_{2k}) \sim \int_0^T P_k\left(\alpha, \log\tfrac{t}{2\pi}\right)\, dt, \tag{7}$$

where

$$P_k(\alpha,x) = \sum_{\sigma\in\Xi} H(\alpha_{\sigma(1)},\dots,\alpha_{\sigma(2k)};x), \tag{8}$$

and Ξ is the set of $\binom{2k}{k}$ permutations such that $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(2k)$. The terms in this set correspond to the number of ways to select, from approximate functional equation for $Z(s)$, the same number k of s's and $(1-s)$'s.

CFKRS, also expressed their sum of $\binom{2k}{k}$ terms as a $2k$-fold residue. We reproduce their second formula for the shifted moments:

$$P_k(\alpha,x) = \frac{(-1)^k}{k!^2}\frac{1}{(2\pi i)^{2k}} \oint\cdots\oint \frac{G(z_1,\dots,z_{2k})\Delta(z_1,\dots,z_{2k})^2}{\displaystyle\prod_{i=1}^{2k}\prod_{j=1}^{2k}(z_i-\alpha_j)} \times e^{\frac{x}{2}\sum_{i=1}^{k} z_i - z_{i+k}}\, dz_1\dots dz_{2k}. \tag{9}$$

While the $\binom{2k}{k}$ terms of the combinatorial sum (8) for $P_k(\alpha,x)$ each have poles of order k^2 at $\alpha=0$, the above is analytic in a neighbourhood of $\alpha=0$ which shows that these poles must cancel. Working with shifts allowed CFKRS to get beyond these poles.

To get formula (2) for $P_k(X)$, set $\alpha=0$ in (9). Even though the formula conjectured by CFKRS is complicated, it does seem to correctly predict the moments. Some evidence, both theoretical and numerical, in its favour was presented in [CFKRS] and [CFKRS2]. For instance, the formulas predicted by CFKRS match with known theorems, including lower terms, for $k=1,2$. CFKRS also computed the moment polynomials for $k\le 7$, and tested the moment conjecture for T roughly of size 10^6.

In this paper we describe a 'cubic accelerant' variant of the second method presented in [CFKRS2] for computing coefficients of the moment polynomials. This allowed us to extend tables of the $c_r(k)$ for $k \leq 13$, and, $0 \leq r \leq k^2$. We also computed many of the coefficients to greater precision. Finally, we used our tables of coefficients to test the moment conjectures up to $k \leq 13$ and T up to 10^8.

Going up to $k = 13$ is substantial, because computing the coefficients to D digits accuracy involves evaluating k^2 sums (one for each r), each sum involving $2k$ choose k terms ($10,400,600$ for $k = 13$), with working precision of roughly $D \times k^2$ digits accuracy. For example, about 2000 digits are required for $k = 13$ with desired precision of 12 digits. The process is made even more challenging, by the fact that each term involves a complicated infinite multivariate product over primes. The RAM requirements are also large because of the number of terms, $\binom{2k}{k}$, that we are computing/updating one prime at a time to high precision.

2 Computing the coefficients of $P_k(x)$

In [CFKRS2], two methods were described for numerically computing the coefficients $c_r(k)$ of the moment polynomials $P_k(x)$.

Their first method involved expanding, in a multivariate Taylor series, the integrand of (2), and working out a technique for expressing the resulting residue (for a general k), giving formulas for $c_r(k)$. For example,

$$\begin{aligned} c_0(k) &= a_k \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} \\ c_1(k) &= c_0(k) 2k^2(\gamma k + B_k(1;)) \\ c_2(k) &= c_0(k)k^2(k-1)(k+1) \\ &\quad \times (2(B_k(1;) + \gamma k)^2 - \gamma^2 - 2\gamma_1 + B_k(1,1;) - B_k(1;1)), \end{aligned} \tag{10}$$

where

$$a_k = \prod_p \left(1 - p^{-1}\right)^{k^2} {}_2F_1(k,k;1;1/p), \tag{11}$$

the γ_j's defined by

$$s\ \zeta(1+s) = 1 + \gamma_0 s - \gamma_1 s^2 + \frac{\gamma_2}{2!}s^3 - \frac{\gamma_3}{3!}s^4 + \cdots, \tag{12}$$

and

$$
\begin{aligned}
B_k(1;) &= \sum_p \frac{k\log(p)}{p-1} - \frac{\log(p)k\,{}_2F_1(k+1,k+1;2;1/p)}{p\,{}_2F_1(k,k;1;1/p)} \qquad (13)\\
B_k(1,1;) &= -\sum_p \left(\frac{\log(p)^2k^2{}_2F_1(k+1,k+1;2;1/p)^2}{p^2\,{}_2F_1(k,k;1;1/p)^2} - \frac{\log(p)^2\binom{k+1}{2}{}_2F_1(k+2,k+2;3;1/p)}{p^2\,{}_2F_1(k,k;1;1/p)} \right)\\
B_k(1;1) &= \sum_p \frac{p\log(p)^2}{(p-1)^2} + \left(\frac{\log(p)^2k^2{}_2F_1(k+1,k+1;2;1/p)^2}{p^2{}_2F_1(k,k;1;1/p)^2} - \right.\\
&\quad \left. \frac{\log(p)^2{}_2F_1(k+1,k+1;1;1/p)}{p\,{}_2F_1(k,k;1;1/p)} \right)\\
B_k(2;) &= -\sum_p \frac{kp\log(p)^2}{(p-1)^2} + \left(\frac{\log(p)^2k^2{}_2F_1(k+1,k+1;2;1/p)^2}{p^2\,{}_2F_1(k,k;1;1/p)^2} - \right.\\
&\quad \left. \frac{\log(p)^2\binom{k+1}{2}\,{}_2F_1(k+2,k+2;3;1/p)}{p^2\,{}_2F_1(k,k;1;1/p)} - \frac{\log(p)^2k\,{}_2F_1(k+2,k+1;2;1/p)}{p\,{}_2F_1(k,k;1;1/p)} \right),
\end{aligned}
$$

with ${}_2F_1$ Gauss' hypergeometric function.

These formulas quickly get much more complicated. In practice CFKRS were able to use this method for $r \le 9$ and compute numerical approximations for all the coefficients of, for example, $P_3(x)$. One advantage of these formulas, expressed as sums over primes, is that one can apply Mobius inversion to accelerate the convergence of these sums, and obtain high precision values of the coefficients.

It soon became apparent [CFKRS] from numerical values of $c_r(k)$ that the leadings coefficients of $P_k(x)$, i.e. associated to the larger powers of x, are very small in comparison to the lower terms. Thus, in order to meaningfully test the moment conjecture for zeta, which involves the moment polynomial evaluated at the slowly growing function $\log(t/2\pi)$ (and this hardly changes over the range of t in which we can gather significant data for $\zeta(1/2+it)$), one needs many coefficients of the moment polynomials. See also [HR] which discusses the uniform asymptotics of these coefficients.

Consequently, a second practical method, relying on the combinatorial sum (8), was developed for computing numerical approximations for *all* k^2 coefficients of the moment polynomial $P_k(x)$.

We detail our computational approach, implementation, and numerical results in the next two sections.

3 Our numerical evaluation of $c_r(k)$

The polynomial $P_k(x)$ given by (2) is the special case $\alpha_1 = \ldots = \alpha_{2k} = 0$ of the function $P_k(\alpha, x)$ in (9). CFRKS's second method for computing the coefficients of $P_k(x)$ relies on their equation (8) for $P_k(\alpha, x)$. However, the terms in (8) have poles if the α_i's are not distinct, coming from the product of k^2 zetas,

$$\prod_{i=1}^{k}\prod_{j=1}^{k}\zeta(1 + z_i - z_{j+k}), \tag{14}$$

that appear in the function G. So we cannot simply substitute $\alpha_j = 0$.

Instead we take the limit as $\alpha_j \to 0$ while making sure that all the α_j's are distinct. Because of the poles, each individual term in (8) becomes very large when α is small, and high precision is needed to see one's way through the resulting cancellation of the poles as we sum across the $\binom{2k}{k}$ terms of the combinatorial sum.

More precisely, consider

$$H(z_1, \ldots, z_{2k}; x) = \exp\left(\frac{x}{2}\sum_{1}^{k} z_j - z_{j+k}\right) A_k(z_1, \ldots, z_{2k}) \prod_{i=1}^{k}\prod_{j=1}^{k}\zeta(1 + z_i - z_{j+k}), \tag{15}$$

and let

$$\epsilon_j = j\epsilon, \tag{16}$$

where $\epsilon \in \mathbb{C}$ is a small number. In practice ϵ was of the form 10^{-D} for some positive integer D.

Using (8) we obtain

$$P_k(x) = \lim_{\epsilon\to 0}\sum_{\sigma\in\Xi} H(\epsilon_{\sigma(1)}, \ldots, \epsilon_{\sigma(2k)}; x). \tag{17}$$

As in [CFKRS2], we expand exp in its Taylor series, and pull out the coefficient of x^{k^2-r}, to get

$$c_r(k) = \frac{1}{2^{k^2-r}(k^2-r)!}\lim_{\epsilon\to 0}\sum_{\sigma\in\Xi} H_r(\epsilon_{\sigma(1)}, \ldots, \epsilon_{\sigma(2k)}), \tag{18}$$

where

$$H_r(z_1, \ldots, z_{2k}) = \left(\sum_{1}^{k} z_j - z_{j+k}\right)^{k^2-r} A_k(z_1, \ldots, z_{2k}) \prod_{i=1}^{k}\prod_{j=1}^{k}\zeta(1 + z_i - z_{j+k}). \tag{19}$$

Notice that, as a function of ϵ, $H_r(\epsilon_{\sigma(1)}, \ldots, \epsilon_{\sigma(2k)})$ has a pole at $\epsilon = 0$ of order r, because the first factor above cancels $k^2 - r$ of the k^2 poles of the double product of zetas. These poles must cancel when summed over permutations σ, otherwise we would not obtain the lhs, $c_r(k)$, as $\epsilon \to 0$. Therefore, because the sum in (18) is analytic about $\epsilon = 0$, we may write

$$c_r(k) = \frac{1}{2^{k^2-r}(k^2-r)!} \left(\left(\sum_{\sigma \in \Xi} H_r(\epsilon_{\sigma(1)}, \ldots, \epsilon_{\sigma(2k)}) \right) + O(|\epsilon|) \right), \tag{20}$$

with the implied constant in the remainder term depending on k and r.

One complication in evaluating the above for a given k and ϵ is that $A_k(z_1, \ldots, z_{2k})$ is expressed as an infinite product over primes as described by (4).

While CFKRS used a 'quadratic accelerant' for evaluating the multivariate Euler product, we implemented a cubic accelerant. This has the advantage of allowing us to truncate the Euler product sooner.

To evaluate $A_k(z_1, \ldots, z_{2k})$, we break up the product over primes into $p \le P$ and $p > P$, where P is a large number. For the first portion $p \le P$, we use the following identity, derived in Section 2.6 of [CFKRS],

$$A_k(z_1, \ldots, z_{2k}) = \prod_p \sum_{j=1}^{k} \prod_{i \ne j}^{k} \frac{\prod_{m=1}^{k} (1 - p^{-1+z_{i+k}-z_m})}{1 - p^{z_{i+k}-z_{j+k}}}, \tag{21}$$

to numerically compute the local factor of $A_k(z_1, \ldots, z_{2k})$ for specific values of p and $z_1, \ldots, z_{2k}$.

Some care must be taken to account for the fact that individual terms in this identity also have poles. While these poles cancel out when summed over j, see the paragraph following equation (2.6.16) in [CFKRS], they cause some additional loss of precision in our application. We are evaluating $A_k(z_1, \ldots, z_{2k})$ at distinct, but small values of z_j. Therefore, when evaluating the sum over j, additional cancellation and hence loss of precision occurs affecting the leading $(k-1)D$ digits, where $\epsilon \approx 10^{-D}$, from the poles of order $k-1$ of the individual terms summed in (21).

For the contribution of the second portion $p > P$, we approximate each local factor appearing in (4) by a product of zeta functions that captures the terms up to degree three in $1/p$ of its multivariate Dirichlet series. A cubic approximation can be obtained by first substituting $u_j = p^{-1/2-z_j}$ and $w_j = p^{-1/2+z_{k+j}}$ into the local factor of (4),

$$\prod_{i,j=1}^{k} (1 - u_i w_j) \int_0^1 \prod_{j=1}^{k} (1 - u_j e(\theta))^{-1} (1 - w_j e(-\theta))^{-1} \, d\theta, \tag{22}$$

and then working out the terms, in the multivariate Maclaurin series, up to degree six, in u_j and w_j.

Notice that the integral over θ pulls out just the terms with the same number of u's and w's. This results in monomials only of even degree appearing. The integral of any other term, which does not have the same number of u's and w's, is zero because it contains a non-zero integer power of $e(\theta)$.

Also, observe that the local factor of (22) is symmetric in the u's and, separately in the w's, meaning if the u_i's are permuted the expression remains invariant and similarly for the w_i's. Also, it is symmetric with u and w, i.e. if all the u's and w's are swapped the expression remains the same.

Therefore, to get terms up to degree six, we can determine the coefficients of representative terms involving u_1, u_2, u_3 and w_1, w_2, w_3, and then symmetrize the resulting expressions over all the u's and w's. More precisely, to get all terms of degrees 2, 4, and 6, it is sufficient to consider only the monomials: u_1w_1, $u_1u_2w_1w_2$, $u_1u_2w_1^2$, $u_1^2w_1^2$, $u_1u_2u_3w_1w_2w_3$, $u_1u_2u_3w_1^2w_2$, $u_1u_2u_3w_1^3$, $u_1^2u_2w_1^2w_2$, $u_1^2u_2w_1^3$, and $u_1^3w_1^3$ instead of every possible monomial, and then exploit symmetry.

Finally, we can simplify further. The integral over θ simply plays the role of pulling out terms with the same number of u's and w's. So, instead of (22), we can work more directly with the function

$$\prod_{i,j=1}^{k}(1-u_iw_j)\prod_{j=1}^{k}(1-u_j)^{-1}(1-w_j)^{-1}. \tag{23}$$

The multivariate Maclaurin series of the above coincides with that of (22) for those terms that have the same number of u's and w's. Furthermore, because we are focusing just on terms involving u_1, u_2, u_3 and w_1, w_2, w_3, we can set $u_j = w_j = 0$ for all $j \geq 4$. Finally, to get terms up to degree six with the same number of u's and w's, we can expand each factor in the denominator as a geometric series of degree 3. We therefore consider:

$$\prod_{i,j=1}^{3}(1-u_iw_j)\prod_{j=1}^{3}(1+u_j+u_j^2+u_j^3)(1+w_j+w_j^2+w_j^3). \tag{24}$$

We expand out the above and tabulate, in Table 1, the coefficients for representative monomials with the same number of u's and w's, up to degree 6, in the multivariate Maclaurin series of the above function, and, hence, equivalently, in (22).

Therefore, symmetrizing, we have that (22) equals

$$1-\sum_{\substack{1\leq i_1<i_2\leq k\\ 1\leq j_1<j_2\leq k}} u_{i_1}u_{i_2}w_{j_1}w_{j_2} + \ 4\sum_{\substack{1\leq i_1<i_2<i_3\leq k\\ 1\leq j_1<j_2<j_3\leq k}} u_{i_1}u_{i_2}u_{i_3}w_{j_1}w_{j_2}w_{j_3}$$

monomial	coefficient
$u_1 w_1$	0
$u_1 u_2 w_1 w_2$	-1
$u_1 u_2 w_1^2$	0
$u_1^2 w_1^2$	0
$u_1 u_2 u_3 w_1 w_2 w_3$	4
$u_1 u_2 u_3 w_1^2 w_2$	1
$u_1 u_2 u_3 w_1^3$	0
$u_1^2 u_2 w_1^2 w_2$	0
$u_1^2 u_2 w_1^3$	0
$u_1^3 w_1^3$	0

Table 1: The second column lists the coefficients that appear with representative monomials, up to degree 6, in the multivariate Maclaurin expansion of (22).

$$+ \sum_{\substack{1\le i_1<i_2<i_3\le k\\ 1\le j_1\ne j_2\le k}} u_{i_1}u_{i_2}u_{i_3}w_{j_1}^2 w_{j_2} + \sum_{\substack{1\le i_1\ne i_2\le k\\ 1\le j_1<j_2<j_3\le k}} u_{i_1}^2 u_{i_2} w_{j_1}w_{j_2}w_{j_3} + \dots \tag{25}$$

Undoing the substitution for u_i's and w_j's, gives the following expansion for the local factors in (4):

$$\begin{aligned} 1 \quad & - \sum_{\substack{1\le i_1<i_2\le k\\ 1\le j_1<j_2\le k}} p^{-2-z_{i_1}-z_{i_2}+z_{k+j_1}+z_{k+j_2}} \\ & + 4 \sum_{\substack{1\le i_1<i_2<i_3\le k\\ 1\le j_1<j_2<j_3\le k}} p^{-3-z_{i_1}-z_{i_2}-z_{i_3}+z_{k+j_1}+z_{k+j_2}+z_{k+j_3}} \\ & \qquad + \sum_{\substack{1\le i_1<i_2<i_3\le k\\ 1\le j_1\ne j_2\le k}} p^{-3-z_{i_1}-z_{i_2}-z_{i_3}+2z_{k+j_1}+z_{k+j_2}} \\ & \qquad\quad + \sum_{\substack{1\le i_1\ne i_2\le k\\ 1\le j_1<j_2<j_3\le k}} p^{-3-2z_{i_1}-z_{i_2}+z_{k+j_1}+z_{k+j_2}+z_{k+j_3}} \\ & \qquad\qquad + \dots, \end{aligned} \tag{26}$$

which we then approximate by the following product:

$$\begin{aligned} & \prod_{\substack{1\le i_1<i_2\le k\\ 1\le j_1<j_2\le k}} \left(1-p^{-2-z_{i_1}-z_{i_2}+z_{k+j_1}+z_{k+j_2}}\right) \\ & \quad \times \prod_{\substack{1\le i_1<i_2<i_3\le k\\ 1\le j_1<j_2<j_3\le k}} \left(1-p^{-3-z_{i_1}-z_{i_2}-z_{i_3}+z_{k+j_1}+z_{k+j_2}+z_{k+j_3}}\right)^{-4} \end{aligned}$$

$$\times \prod_{\substack{1\le i_1<i_2<i_3\le k\\ 1\le j_1\ne j_2\le k}} (1-p^{-3-z_{i_1}-z_{i_2}-z_{i_3}+2z_{k+j_1}+z_{k+j_2}})^{-1}$$
$$\times \prod_{\substack{1\le i_1\ne i_2\le k\\ 1\le j_1<j_2<j_3\le k}} (1-p^{-3-2z_{i_1}-z_{i_2}+z_{k+j_1}+z_{k+j_2}+z_{k+j_3}})^{-1}. \tag{27}$$

The last step can be seen by expanding each factor in a geometric series and comparing the terms, up to those containing a $1/p^3$, with those in (26). We also remark that, had we wanted a quartic approximation, then slightly more care would be needed as the first product above would, on expanding in geometric series, interact with the quartic terms.

Note that, when all $z_j = 0$, the above gives an approximation for the local factor with a remainder term of size $O_k(p^{-4})$. Hence, the accumulated remainder for $p > P$ in this approximation, when $z_j = 0$, is

$$\prod_{p>P}(1+O(p^{-4})) \ll \sum_{p>P} p^{-4} \ll \log(P)^{-1}P^{-3}, \tag{28}$$

with the implied constant depending on k.

The product in (27) allows us to approximate the tail, i.e. for $p > P$, of $A_k(z_1,\ldots,z_k)$ in terms of zeta:

$$\prod_{p>P}\prod_{i,j=1}^{k}(1-p^{-1-z_i+z_{k+j}})\int_0^1 \prod_{j=1}^{k}\left(1-\frac{e(\theta)}{p^{\frac12+z_j}}\right)^{-1}\left(1-\frac{e(-\theta)}{p^{\frac12-z_{k+j}}}\right)^{-1} d\theta$$
$$\approx \frac{\prod_{\substack{1\le i_1<i_2\le k\\ 1\le j_1<j_2\le k}} \zeta(2+z_{i_1}+z_{i_2}-z_{k+j_1}-z_{k+j_2})^{-1}}{\prod_{p\le P}\prod_{\substack{1\le i_1<i_2\le k\\ 1\le j_1<j_2\le k}}(1-p^{-2-z_{i_1}-z_{i_2}+z_{k+j_1}+z_{k+j_2}})}$$
$$\times\frac{\prod_{\substack{1\le i_1<i_2<i_3\le k\\ 1\le j_1<j_2<j_3\le k}} \zeta(3+z_{i_1}+z_{i_2}+z_{i_3}-z_{k+j_1}-z_{k+j_2}-z_{k+j_3})^{4}}{\prod_{p\le P}\prod_{\substack{1\le i_1<i_2<i_3\le k\\ 1\le j_1<j_2<j_3\le k}}(1-p^{-3-z_{i_1}-z_{i_2}-z_{i_3}+z_{k+j_1}+z_{k+j_2}+z_{k+j_3}})^{-4}}$$
$$\times\frac{\prod_{\substack{1\le i_1<i_2<i_3\le k\\ 1\le j_1\ne j_2\le k}} \zeta(3+z_{i_1}+z_{i_2}+z_{i_3}-2z_{k+j_1}-z_{k+j_2})}{\prod_{p\le P}\prod_{\substack{1\le i_1<i_2<i_3\le k\\ 1\le j_1\ne j_2\le k}}(1-p^{-3-z_{i_1}-z_{i_2}-z_{i_3}+2z_{k+j_1}+z_{k+j_2}})^{-1}} \tag{29}$$
$$\times\frac{\prod_{\substack{1\le i_1\ne i_2\le k\\ 1\le j_1<j_2<j_3\le k}} \zeta(3+2z_{i_1}+z_{i_2}-z_{k+j_1}-z_{k+j_2}-z_{k+j_3})}{\prod_{p\le P}\prod_{\substack{1\le i_1\ne i_2\le k\\ 1\le j_1<j_2<j_3\le k}}(1-p^{-3-2z_{i_1}-z_{i_2}+z_{k+j_1}+z_{k+j_2}+z_{k+j_3}})^{-1}}.$$

4 Implementation and tables of coefficients $c_r(k)$

Our code was implemented in `C++` using the GNU MPFR library [FHLPZ], along with Jon Wilkening's `C++` wrapper for MPFR. MPFR is based on GMP, the GNU multiprecision library. We used gcc, the GNU C compiler, to compile our code, with the '-fopenmp' option in order to enable the use of OpenMP directives in our code. This allowed us to carry out some of the key steps in parallel for a given k, using several cores of our machine.

For each k, we selected a precision, specified by the number of digits desired, 'Digits', for the final output, and let $\epsilon = 10^{-\text{Digits}}$. For example, we used Digits$= 25$, i.e. $\epsilon = 10^{-25}$, for $k = 4$. We then put $\epsilon_j = j\epsilon$, $1 \le j \le k$, and set about computing the sum (20), using our cubic multivariate approximation for the tail of the infinite product, i.e. equation (29).

Observe, in (19), that the dependence on r manifests only at the factor:

$$\left(\sum_1^k z_j - z_{j+k} \right)^{k^2 - r}. \tag{30}$$

Therefore, we were able to store and recycle all the other quantities across $0 \le r \le k^2$.

We record one important hack that we used several times in our program. While the double product of zetas in (19) involves k^2 factors, many of these are repeated since there are just $4k-2$ possible values of $\epsilon_a - \epsilon_b = (a-b)\epsilon$, where a, b are distinct integers in $[1, 2k]$. The same holds, for each p, in the double products in (21).

Likewise, while the products in (29) involve up to $O(k^6)$ factors, these appear with multiplicity, and there are just $O(k)$ distinct factors. This is true both for the product of zetas in the numerator, and also, for each p, the factors that appear in the denominator.

We exploited these multiplicities by computing and storing a table of the distinct values of zeta that appear and, for each p, of the distinct factors that occur. Furthermore, we took advantage of the fact that the powers of p that occur, other than $1/p$, are of the form $p^{m\epsilon}$, where $m \in \mathbb{Z}$, and thus computed and stored them by repeated multiplication of p^ϵ and of $1/p^\epsilon$.

To account for the high amount of cancellation that occurs as a consequence of the poles of the individual terms in (20), we let our working precision be equal to

$$\text{WorkingDigits} = (k^2 + k - 1 + 6)\text{Digits}, \tag{31}$$

and carried out our computations using these many digits. The k^2 was to account for the largest order poles occurring in H_r, when $r = k^2$, of order k^2. While we could

have gotten away with less precision for smaller r, we recycled most of the computed quantities across all r. The $k-1$ acounts for cancellation amongst the poles of the terms in the sum over j in (21). Finally we needed to have some working precision left over, after all the cancellation, to capture $c_r(k)$ to Digits precision. The +6Digits was chosen to give us some leeway. For example, we had WorkingDigits= 625 for $k=4$ and Digits= 25, and WorkingDigits= 2244 for $k=13$ and Digits= 12.

Note that using a specific desired precision does not necessarily result in that precision being achieved as one also needs to take into account the implied constant in the O term in (20) which depends both on k and on r, and the fact that it becomes difficult for larger k, even with our cubic accelerant, to take sufficiently many primes p in the multivariate Euler product A_k. The bound (28) provides a rough estimate for the relative error introduced by truncating the Euler product, using our cubic accelerant at a given P, and very small ϵ (strictly speaking, the remainder term in (28) needs to have the P^{-3} replaced by $P^{-3+8k\epsilon}$ to take into account the choice of ϵ_j). Therefore, $P \gg_k 10^{-\text{Digits}/3}$ is a crude estimate for how many primes are needed in the Euler product to achieve a relative error smaller than $10^{-\text{Digits}}$.

Rather than working with explicit constants in the truncation bounds, both for the O term in (20) and the Euler product, we experimented by taking different values of $\epsilon = 10^{-\text{Digits}}$ and P, using our estimates as guides. We inspected the stability of our numerical values of $c_r(k)$ by comparing those computed for a given P against those with P replaced by the first prime smaller than $P/3$, and only outputting the digits that agreed. It seems, from our tables, that the coefficients $c_r(k)$ with mid-range values of r are more stable and converge faster with respect to P, especially for larger values of k. We did not explore the reason for this, but presumably the lower terms, beyond those resulting from our cubic accelerant, have comparatively smaller coefficients for those values of r for which $c_r(k)$ converges faster.

Numerical values of the coefficients for $4 \le k \le 13$ thus obtained are presented in the Tables 2- 9 below in scientific 'e' notation, for example $1.2e-3 = 1.2 \times 10^{-3}$. High precision values of $c_r(k)$ for $k = 1, 2, 3$, can be obtained from [CFKRS].

It is also worth mentioning that all the digits of the coefficients computed in this manner agree, except in a few instances where the last decimal place differs slightly, with the results of the first method of [CFKRS2] (see Section 2). That method has the advantage of producing high precision values of the coefficients, but is limited to relatively small values of r. We reran the program used in [CFKRS2] for $k \le 13$ and $r \le 7$ and display those values in Table 10 for comparison.

r	$c_r(4)$	$c_r(5)$	$c_r(6)$
0	2.4650183919342273540798944e-13	1.4160010206227312009555e-24	5.12947340914919112e-40
1	5.4501405731171865593630583e-11	7.3804127564944513059668e-22	5.30673280992644424 6e-37
2	5.2877296347912031138489 7e-09	1.7797796235196529053094e-19	2.6079207711483512396e-34
3	2.9641143179993979459691 8e-07	2.6358866096607247582 86e-17	8.1016132157790177281e-32
4	1.0645950068128470513211 82e-05	2.6840545349997485760134e-15	1.7861297380093099773e-29
5	2.5702983342426340235494e-04	1.9936413092498971803 12e-13	2.9743167108636063482e-27
6	4.2639216163116947218762e-03	1.1184855124933629437778e-11	3.8877082911558678876e-25
7	4.8941424514216010271276 1e-02	4.842797553044804165519e-10	4.0922426140686293551 4e-23
8	3.8785266540195534998 33e-01	1.6398013084961560997 97e-08	3.5314663856570325725e-21
9	2.1091338286487335520 4e+00	4.3749351054922463304 12e-07	2.5306376900609734782 89e-19
10	7.8325356118822623579303e+00	9.2263335029653032633 7e-06	1.5198191029685924995e-17
11	1.982806812499890923e+01	1.5376777820710794699 1e-04	7.7001513760923745827 0e-16
12	3.388893203738368856e+01	2.0190277580781319590 7e-03	3.30612104141074360 46e-14
13	3.82033062189019517e+01	2.077270672848464754 74e-02	1.2064041518984715612e-12
14	2.560441501227035e+01	1.6625058643910393652e-01	3.7467192541626917996e-11
15	1.06189693794016e+01	1.0264667778494737 56e+00	9.9056942856889097902e-10
16	7.089464552244e-01	4.8485892783436424 78e+00	2.2273885767179683823e-08
17		1.73908760901310234e+01	4.2513728668167860 76e-07
18		4.7040877087561734e+01	6.8674335769870947550e-06
19		9.511661794587886e+01	9.3515830187750442 62e-05
20		1.41444460064317e+02	1.0683164211730225 28e-03
21		1.4935694999630e+02	1.018070238623614 85e-02
22		1.0588728028422e+02	8.041867930583792 44e-02
23		4.41362307288e+01	5.2296141941724947e-01
24		2.010650046e+01	2.7802017665195719e+00
25		-1.2701703e+00	1.200111408801811e+01
26			4.179670936891264e+01
27			1.16723095829484e+02
28			2.5939897299715e+02
29			4.524908135220e+02
30			6.0117334836510e+02
31			5.7354384553122e+02
32			3.75018676133e+02
33			2.46890415605e+02
34			2.454954369e+02
35			1.603303769e+02
36			-3.78219665e+01

Table 2: Coefficients $c_r(k)$ for $k = 4, 5, 6$. For $k = 4, 5$ we used $\epsilon = 10^{-25}$, and $P = 942939827, 180343651$ respectively. For $k = 6$ we combined two data sets using $\epsilon = 10^{-25}$, $P = 25501199$ for $0 \le r \le 29$, and $\epsilon = 10^{-16}$, $P = 608121859$ for $30 \le r \le 36$.

r	$c_r(7)$	r	$c_r(7)$	r	$c_r(7)$
0	6.5822847876005500e-60	1	1.2041430555451870e-56	2	1.0621355717492720e-53
3	6.0172653760159300e-51	4	2.4606287673240130e-48	5	7.7390121665211530e-46
6	1.9478649494952360e-43	7	4.0307684926363700e-41	8	6.9917763337237880e-39
9	1.0314019779812270e-36	10	1.3082869144993580e-34	11	1.4392681201435320e-32
12	1.3825312154986080e-30	13	1.1657759371318020e-28	14	8.6652476933527220e-27
15	5.6962227424753780e-25	16	3.3197648540507990e-23	17	1.7183970393294220e-21
18	7.9096788893235440e-20	19	3.2396929335740840e-18	20	1.1809579273268370e-16
21	3.8302270051510130e-15	22	1.1044706290361260e-13	23	2.8282258231583490e-12
24	6.4210662257609940e-11	25	1.2898755567219640e-09	26	2.2869667400876520e-08
27	3.5683995004969530e-07	28	4.8834071041615500e-06	29	5.8391045220798220e-05
30	6.0742037327532430e-04	31	5.4716438254364890e-03	32	4.2465903403750590e-02
33	2.824549346789606e-01	34	1.601333066518585e+00	35	7.696699614092694e+00
36	3.12035202072577e+01	37	1.06197143546798e+02	38	3.019136554174e+02
39	7.117410357280e+02	40	1.37009445510e+03	41	2.0827987442e+03
42	2.357363536e+03	43	1.93463843e+03	44	1.75714310e+03
45	2.853378e+03	46	3.100593e+03	47	3.3940e+02
48	-1.20854e+03	49	-5.0194e+02		

Table 3: Coefficients for $k = 7$ truncating at $P = 11015647$, and using $\epsilon = 10^{-16}$.

r	$c_r(8)$	r	$c_r(8)$	r	$c_r(8)$
0	1.870442160117e-84	1	5.570219365179e-81	2	8.0727983790767e-78
3	7.5876025208718e-75	4	5.2002464291967e-72	5	2.7705098412043e-69
6	1.1944832708049e-66	7	4.28399526987474e-64	8	1.30388783552972e-61
9	3.41901547588078e-59	10	7.814883564309408e-57	11	1.5716390984187150e-54
12	2.8019878105779590e-52	13	4.455999566980015e-50	14	6.353422665784412e-48
15	8.1564318452061200e-46	16	9.461598292152723e-44	17	9.946939863498895e-42
18	9.5005755608004230e-40	19	8.2610748506972050e-38	20	6.5505802739681480e-36
21	4.7431624904297110e-34	22	3.1395134797176380e-32	23	1.9011136618426250e-30
24	1.0537646192031000e-28	25	5.3481979466238930e-27	26	2.4856705239072160e-25
27	1.0578033941072590e-23	28	4.1205620801239040e-22	29	1.4685071632966620e-20
30	4.7847077672871950e-19	31	1.4239651481852300e-17	32	3.8665596212165030e-16
33	9.566613928613862e-15	34	2.1534462266724710e-13	35	4.4024065808181710e-12
36	8.1576484690923080e-11	37	1.367077736688555e-09	38	2.0668154599686450e-08
39	2.811291057443101e-07	40	3.430097716487554e-06	41	3.741873571285350e-05
42	3.63684791980377e-04	43	3.137462556407600e-03	44	2.39287182393225e-02
45	1.60675070999268e-01	46	9.4588131743684e-01	47	4.8616355021107e+00
48	2.173103986560e+01	49	8.417133459453e+01	50	2.81517268211e+02
51	8.0929177383e+02	52	1.9821841216e+03	53	4.05873574e+03
54	6.69566487e+03	55	8.4203977e+03	56	8.096360e+03
57	9.4961243e+03	58	1.99106e+04	59	3.09087e+04
60	1.3133e+04	61	-2.964e+04	62	-4.0582e+04
63	-8.56e+03	64	4.56e+03		

Table 4: Coefficients for $k = 8$ truncating at $P = 1212569$, and using $\epsilon = 10^{-16}$.

r	$c_r(9)$	r	$c_r(9)$	r	$c_r(9)$
0	7.920155238e-114	1	3.608743873e-110	2	8.051296272e-107
3	1.172362406e-103	4	1.2530058769e-100	5	1.0481427376e-97
6	7.1456310032e-95	7	4.0821744596e-92	8	1.9941925290e-89
9	8.4594205088e-87	10	3.1538144173e-84	11	1.0433789942e-81
12	3.08731306439e-79	13	8.22414128961e-77	14	1.98314875716e-74
15	4.34906593630e-72	16	8.70843432097e-70	17	1.597618788502e-67
18	2.693264006511e-65	19	4.1828312947266e-63	20	5.9981084135426e-61
21	7.9570365567690e-59	22	9.7816383607331e-57	23	1.1159063652824e-54
24	1.1828936996292e-52	25	1.16636205776320e-50	26	1.0707494871812e-48
27	9.1588630961799e-47	28	7.30410594473147e-45	29	5.43352058419794e-43
30	3.77180308962311e-41	31	2.44391067332666e-39	32	1.47828492081742e-37
33	8.34814115852835e-36	34	4.401070135763961e-34	35	2.16570032669324e-32
36	9.94493396339446e-31	37	4.26008480938235e-29	38	1.7015865979528490e-27
39	6.33392559975936e-26	40	2.19581041600330e-24	41	7.08426675189073e-23
42	2.1252127802020740e-21	43	5.92241988702831e-20	44	1.53150283306451e-18
45	3.670620764266784e-17	46	8.14314143690125e-16	47	1.66973151449710e-14
48	3.15948278435009e-13	49	5.507449862131347e-12	50	8.82746086945955e-11
51	1.29834184462004e-09	52	1.74846308044295e-08	53	2.1508513613974e-07
54	2.4107241046116e-06	55	2.4552071751795e-05	56	2.265572158192e-04
57	1.888394621727e-03	58	1.4172609520338e-02	59	9.546112504952e-02
60	5.751640191248e-01	61	3.08994036050e+00	62	1.47569216973e+01
63	6.2480977623e+01	64	2.3393469844e+02	65	7.721276404e+02
66	2.23430645e+03	67	5.60365351e+03	68	1.1907448e+04
69	2.062257e+04	70	2.776900e+04	71	3.06185e+04
72	4.7187e+04	73	1.2202e+05	74	2.314e+05
75	1.255e+05	76	-4.658e+05	77	-1.07e+06
78	-5.794e+05	79	6.75e+05	80	8.27e+05
81	1.3e+05				

Table 5: Coefficients for $k = 9$ truncating at $P = 170741$, and using $\epsilon = 10^{-16}$.

r	$c_r(10)$	r	$c_r(10)$	r	$c_r(10)$
0	3.54888492477e-148	1	2.35769133101e-144	2	7.70233663026e-141
3	1.64948634407e-137	4	2.60451944693e-134	5	3.23366677841e-131
6	3.28765141574e-128	7	2.81472946999e-125	8	2.07111222708e-122
9	1.330223430450e-119	10	7.54902089850e-117	11	3.822610704580e-114
12	1.741117024160e-111	13	7.181397221310e-109	14	2.697527199940e-106
15	9.272621188300e-104	16	2.929091296840e-101	17	8.533614631040e-99
18	2.300282495690e-96	19	5.752933603890e-94	20	1.338222245340e-91
21	2.901664262540e-89	22	5.876116434230e-87	23	1.113295629950e-84
24	1.976420578280e-82	25	3.292284406610e-80	26	5.152298303310e-78
27	7.583498206660e-76	28	1.050825438590e-73	29	1.372032297570e-71
30	1.689305278750e-69	31	1.962725749490e-67	32	2.153176878340e-65
33	2.231491144780e-63	34	2.185751487190e-61	35	2.024236345360e-59
36	1.773015880790e-57	37	1.469142759970e-55	38	1.151856372240e-53
39	8.546210010780e-52	40	6.000996648940e-50	41	3.988024973250e-48
42	2.508210346100e-46	43	1.492814370840e-44	44	8.406690331770e-43
45	4.478585581910e-41	46	2.256576452310e-39	47	1.075045360230e-37
48	4.840860160570e-36	49	2.059518250610e-34	50	8.274865115540e-33
51	3.138261284280e-31	52	1.122809702480e-29	53	3.787412159890e-28
54	1.203656074660e-26	55	3.601325369860e-25	56	1.013609245800e-23
57	2.681299896780e-22	58	6.659997348320e-21	59	1.551711921570e-19
60	3.387479938260e-18	61	6.920780548090e-17	62	1.321576287680e-15
63	2.355560477620e-14	64	3.913136454580e-13	65	6.049282423090e-12
66	8.687675538270e-11	67	1.157038978680e-09	68	1.426296610210e-08
69	1.624086810140e-07	70	1.704560517010e-06	71	1.645250259970e-05
72	1.456896715700e-04	73	1.180639233900e-03	74	8.733236736800e-03
75	5.881138307580e-02	76	3.596179964360e-01	77	1.991704581680e+00
78	9.96798553888e+00	79	4.49881033284e+01	80	1.8275974779e+02
81	6.6682890321e+02	82	2.177262632e+03	83	6.31448075e+03
84	1.60290666e+04	85	3.47003569e+04	86	6.163495e+04
87	8.726963e+04	88	1.147239e+05	89	2.48873e+05
90	7.40512e+05	91	1.4296e+06	92	2.56e+05
93	-6.2748e+06	94	-1.489e+07	95	-7.97e+06
96	2.260e+07	97	4.02e+07	98	1.43e+07
99	-1.08e+07	100	-5.22e+06		

Table 6: Coefficients for $k = 10$ truncating at $P = 675929$, and using $\epsilon = 10^{-16}$.

r	$c_r(11)$	r	$c_r(11)$	r	$c_r(11)$	r	$c_r(11)$
0	1.2451314e-187	1	1.16057289e-183	2	5.33593693e-180	3	1.61328064e-176
4	3.60796892e-173	5	6.36556626e-170	6	9.22782529e-167	7	1.13037084e-163
8	1.19424156e-160	9	1.10531689e-157	10	9.07262368e-155	11	6.67001789e-152
12	4.42795938e-149	13	2.67250329e-146	14	1.47494511e-143	15	7.48038633e-141
16	3.50123968e-138	17	1.51807108e-135	18	6.117325360e-133	19	2.297695573e-130
20	8.065058896e-128	21	2.651612616e-125	22	8.182770981e-123	23	2.374593250e-120
24	6.490958842e-118	25	1.673872653e-115	26	4.077854544e-113	27	9.396885825e-111
28	2.050582239e-108	29	4.2419484994e-106	30	8.3264998645e-104	31	1.5521939465e-101
32	2.7501740419e-99	33	4.6346698213e-97	34	7.4337414866e-95	35	1.13549625358e-92
36	1.65268057424e-90	37	2.29313645275e-88	38	3.03459182593e-86	39	3.831526640190e-84
40	4.617401084980e-82	41	5.312669020390e-80	42	5.837617513170e-78	43	6.127289274640e-76
44	6.144657440350e-74	45	5.888372616130e-72	46	5.392853053390e-70	47	4.720750019910e-68
48	3.950047958650e-66	49	3.159433634290e-64	50	2.415656293490e-62	51	1.765512476040e-60
52	1.233372385020e-58	53	8.235138259720e-57	54	5.254770626530e-55	55	3.203931860340e-53
56	1.866325220220e-51	57	1.038439545240e-49	58	5.517819788010e-48	59	2.799212525590e-46
60	1.355385595070e-44	61	6.261980454760e-43	62	2.759510005390e-41	63	1.159469298800e-39
64	4.643172007710e-38	65	1.771352314140e-36	66	6.434590479390e-35	67	2.224528434850e-33
68	7.315010837230e-32	69	2.286618699140e-30	70	6.790446084110e-29	71	1.914406176510e-27
72	5.120202491620e-26	73	1.298144910520e-24	74	3.117357525500e-23	75	7.084341566220e-22
76	1.522160095850e-20	77	3.089177852240e-19	78	5.915547597220e-18	79	1.067669213960e-16
80	1.814092977890e-15	81	2.898172629520e-14	82	4.347699833250e-13	83	6.115899813510e-12
84	8.055381828660e-11	85	9.918802503370e-10	86	1.139891722860e-08	87	1.220516286130e-07
88	1.215356440380e-06	89	1.123334898600e-05	90	9.618099818200e-05	91	7.612754647750e-04
92	5.558326140280e-03	93	3.73566507420e-02	94	2.3062374960e-01	95	1.3052451738e+00
96	6.7601261642e+00	97	3.199030312e+01	98	1.381287107e+02	99	5.43353076e+02
100	1.94226636e+03	101	6.2767681e+03	102	1.8153278e+04	103	4.6140024e+04
104	1.001842e+05	105	1.79901e+05	106	2.732785e+05	107	4.8073e+05
108	1.4164e+06	109	4.18178e+06	110	6.523e+06	111	-7.31e+06
112	-6.295e+07	113	-1.26e+08	114	-1.2e+07	115	4.21e+08
116	7.32e+08	117	1.7e+08	118	-8.1e+08	119	-8.2e+08
120	-1.1e+08	121	9.9e+07				

Table 7: Coefficients for $k = 11$ truncating at $P = 85889$, and using $\epsilon = 10^{-12}$.

r	$c_r(12)$	r	$c_r(12)$	r	$c_r(12)$	r	$c_r(12)$
0	2.61438e-232	1	3.31314e-228	2	2.07583e-224	3	8.57284e-221
4	2.62512e-217	5	6.35698e-214	6	1.26799e-210	7	2.14259e-207
8	3.13061e-204	9	4.01773e-201	10	4.58505e-198	11	4.69936e-195
12	4.36135e-192	13	3.69038e-189	14	2.86364e-186	15	2.04802e-183
16	1.35583e-180	17	8.34021e-178	18	4.78307e-175	19	2.56498e-172
20	1.28962e-169	21	6.09352e-167	22	2.71168e-164	23	1.13871e-161
24	4.52018e-159	25	1.698887e-156	26	6.054452e-154	27	2.048658e-151
28	6.589967e-149	29	2.017472e-146	30	5.884306e-144	31	1.636682e-141
32	4.345107e-139	33	1.101947e-136	34	2.671618e-134	35	6.196518e-132
36	1.375825e-129	37	2.926060e-127	38	5.964172e-125	39	1.165705e-122
40	2.185779e-120	41	3.9336446e-118	42	6.7972368e-116	43	1.1281875e-113
44	1.7992516e-111	45	2.7580481e-109	46	4.0647787e-107	47	5.7611834e-105
48	7.8547080e-103	49	1.0303518e-100	50	1.3006541e-98	51	1.58027693e-96
52	1.84826712e-94	53	2.08119455e-92	54	2.25644465e-90	55	2.35580895e-88
56	2.36859061e-86	57	2.293494547e-84	58	2.138841512e-82	59	1.921057763e-80
60	1.6618175746e-78	61	1.3845226981e-76	62	1.110902429e-74	63	8.583981848e-73
64	6.3871821912e-71	65	4.5761550941e-69	66	3.1565961813e-67	67	2.09610369144e-65
68	1.33974800679e-63	69	8.2410737927e-62	70	4.8777660410e-60	71	2.7774938619e-58
72	1.5212162503e-56	73	8.01193333444e-55	74	4.05683148804e-53	75	1.97436017587e-51
76	9.23280422076e-50	77	4.14742073168e-48	78	1.78904091502e-46	79	7.40817919355e-45
80	2.943700133510e-43	81	1.122014888380e-41	82	4.10061125513e-40	83	1.43633249117e-38
84	4.81968183211e-37	85	1.548559986510e-35	86	4.76169394977e-34	87	1.400502643990e-32
88	3.93774707023e-31	89	1.05777224840e-29	90	2.71295150626e-28	91	6.63907833056e-27
92	1.549117947290e-25	93	3.44390394051e-24	94	7.28902479104e-23	95	1.467519582110e-21
96	2.808147993620e-20	97	5.10250621963e-19	98	8.79548303360e-18	99	1.43685110544e-16
100	2.22218436508e-15	101	3.24998071837e-14	102	4.489567612890e-13	103	5.85079932579e-12
104	7.18370974878e-11	105	8.29873790933e-10	106	9.0070339192e-09	107	9.1707380762e-08
108	8.7457189835e-07	109	7.7990666196e-06	110	6.4924284455e-05	111	5.036525894e-04
112	3.6345151422e-03	113	2.4355146991e-02	114	1.5129598801e-01	115	8.699064420e-01
116	4.62298136e+00	117	2.268127865e+01	118	1.02629299e+02	119	4.2781349e+02
120	1.6399167e+03	121	5.7590488e+03	122	1.839118e+04	123	5.26980e+04
124	1.3266774e+05	125	2.8583e+05	126	5.2381e+05	127	9.3264e+05
128	2.342e+06	129	7.741e+06	130	1.944e+07	131	1.41e+07
132	-1.08e+08	133	-4.566e+08	134	-6.14e+08	135	1.1e+09
136	5.58e+09	137	7.3e+09	138	-6.6e+09	139	-3.37e+10
140	-3.7e+10	141	5.5e+09	142	4.3e+10	143	2.7e+10
144	1.9e+09						

Table 8: Coefficients for $k = 12$ truncating at $P = 12979$, and using $\epsilon = 10^{-12}$.

r	$c_r(13)$	r	$c_r(13)$	r	$c_r(13)$	r	$c_r(13)$
0	2.58e-282	1	4.33e-278	2	3.60e-274	3	1.97e-270
4	8.05e-267	5	2.60e-263	6	6.94e-260	7	1.57e-256
8	3.08e-253	9	5.31e-250	10	8.16e-247	11	1.13e-243
12	1.42e-240	13	1.63e-237	14	1.71e-234	15	1.67e-231
16	1.51e-228	17	1.27e-225	18	9.96e-223	19	7.34e-220
20	5.08e-217	21	3.31e-214	22	2.04e-211	23	1.19e-208
24	6.56e-206	25	3.43e-203	26	1.71e-200	27	8.11e-198
28	3.67e-195	29	1.58e-192	30	6.52e-190	31	2.57e-187
32	9.69e-185	33	3.50e-182	34	1.21e-179	35	4.04e-177
36	1.29e-174	37	3.96e-172	38	1.17e-169	39	3.32e-167
40	9.068e-165	41	2.387e-162	42	6.054e-160	43	1.480e-157
44	3.490e-155	45	7.939e-153	46	1.743e-150	47	3.695e-148
48	7.563e-146	49	1.496e-143	50	2.858e-141	51	5.280e-139
52	9.429e-137	53	1.628e-134	54	2.720e-132	55	4.396e-130
56	6.874e-128	57	1.040e-125	58	1.524e-123	59	2.160e-121
60	2.966e-119	61	3.942e-117	62	5.074e-115	63	6.3253e-113
64	7.6365e-111	65	8.9299e-109	66	1.0115e-106	67	1.1098e-104
68	1.1795e-102	69	1.2144e-100	70	1.2112e-98	71	1.1702e-96
72	1.0952e-94	73	9.9295e-93	74	8.7200e-91	75	7.4173e-89
76	6.11074e-87	77	4.87569e-85	78	3.76736e-83	79	2.81878e-81
80	2.04204e-79	81	1.43219e-77	82	9.72340e-76	83	6.38944e-74
84	4.063248e-72	85	2.500254e-70	86	1.488419e-68	87	8.5708030e-67
88	4.7730131e-65	89	2.5701193e-63	90	1.3378668e-61	91	6.7309343e-60
92	3.2721950e-58	93	1.5367284e-56	94	6.9700536e-55	95	3.0523694e-53
96	1.29025276e-51	97	5.26280865e-50	98	2.0707478e-48	99	7.8570340e-47
100	2.8738212e-45	101	1.01291060e-43	102	3.43894917e-42	103	1.124213471e-40
104	3.53717984e-39	105	1.07067892e-37	106	3.11641068e-36	107	8.71833921e-35
108	2.34302639e-33	109	6.04582505e-32	110	1.49702508e-30	111	3.55505913e-29
112	8.09182488e-28	113	1.76422359e-26	114	3.68197859e-25	115	7.35070729e-24
116	1.40275907e-22	117	2.55690377e-21	118	4.44814493e-20	119	7.37932244e-19
120	1.16640624e-17	121	1.755027226e-16	122	2.51133489e-15	123	3.41411712e-14
124	4.40504845e-13	125	5.38821792e-12	126	6.24112402e-11	127	6.83724634e-10
128	7.07544905e-09	129	6.90733979e-08	130	6.352697060e-07	131	5.49643677e-06
132	4.4673281e-05	133	3.40576382e-04	134	2.4318473e-03	135	1.6239805e-02
136	1.0128549e-01	137	5.8923590e-01	138	3.194017e+00	139	1.611797e+01
140	7.566471e+01	141	3.301742e+02	142	1.337419e+03	143	5.014434e+03
144	1.73025e+04	145	5.43813e+04	146	1.53198e+05	147	3.7902e+05
148	8.147e+05	149	1.6140e+06	150	3.743e+06	151	1.18e+07
152	3.577e+07	153	6.04e+07	154	-8.98e+07	155	-8.78e+08
156	-2.1e+09	157	1.30e+09	158	2.1e+10	159	5.2e+10
160	-9e+09	161	-3.6e+11	162	-8.7e+11	163	-5.2e+11
164	1.6e+12	165	3.83e+12	166	2.6e+12	167	-9e+11
168	-1.65e+12	169	-3.7e+11				

Table 9: Coefficients for $k = 13$ truncating at $P = 1699$, and using $\epsilon = 10^{-12}$.

k	r	$c_r(k)$	k	r	$c_r(k)$
4	0	.24650183919342273540799e-12	9	0	.79201552383685290316e-113
4	1	.545014057311718655936e-10	9	1	.36087438729455558616e-109
4	2	.5287729634791203113849e-8	9	2	.80512962716760934894e-106
4	3	.29641143179993979459691e-6	9	3	.11723624058166636900e-102
4	4	.10645950068128470513211e-4	9	4	.12530058768923713471e-99
4	5	.2570298334242634023549e-3	9	5	.10481427375523351016e-96
4	6	.426392161631169472187e-2	9	6	.71456310032205157639e-94
4	7	.4894142451421601027126e-1	9	7	.40821744596370636463e-91
5	0	.14160010206227312010e-23	10	0	.35488849247730348098e-147
5	1	.73804127564944513060e-21	10	1	.23576913310137009644e-143
5	2	.17797796235196529053e-18	10	2	.77023366302575780180e-140
5	3	.26358866096607247583e-16	10	3	.16494863440733411303e-136
5	4	.26840545349997485760e-14	10	4	.26045194469316625626e-133
5	5	.19936413092498971803e-12	10	5	.32336677784065596864e-130
5	6	.11184855124933629438e-10	10	6	.32876514157441589044e-127
5	7	.48427975530448041655e-9	10	7	.28147294699934449064e-124
6	0	.51294734091491911243e-39	11	0	.124513138816594309e-186
6	1	.53067328099264442456e-36	11	1	.116057289076806867e-182
6	2	.26079207711483512396e-33	11	2	.533593692953085661e-179
6	3	.81016132157790177281e-31	11	3	.161328064239033845e-175
6	4	.17861297380093099773e-28	11	4	.360796891855797563e-172
6	5	.29743167108636063482e-26	11	5	.636556626245602757e-169
6	6	.38877082911558678876e-24	11	6	.922782528634884471e-166
6	7	.40922426140686293551e-22	11	7	.113037084302453487e-162
7	0	.65822847876005499378e-59	12	0	.26143756530064042e-231
7	1	.12041430555451865785e-55	12	1	.33131354510381580e-227
7	2	.10621355717492716606e-52	12	2	.20758311485643071e-223
7	3	.60172653760159300486e-50	12	3	.85728395653185504e-220
7	4	.24606287673240130820e-47	12	4	.26251165314413802e-216
7	5	.77390121665211526042e-45	12	5	.63569771378458374e-213
7	6	.19478649494952357456e-42	12	6	.12679941162107841e-209
7	7	.40307684926363697065e-40	12	7	.21425932836667245e-206
8	0	.18704421601168844202e-83	13	0	.2577425553942569e-281
8	1	.55702193651787573285e-80	13	1	.4326313738224894e-277
8	2	.80727983790767114280e-77	13	2	.3596648214485737e-273
8	3	.75876025208717340817e-74	13	3	.1974403388369282e-269
8	4	.52002464291967325362e-71	13	4	.8051097934153342e-266
8	5	.27705098412043360903e-68	13	5	.2601086374395006e-262
8	6	.11944832708048773794e-65	13	6	.6934795975034907e-259
8	7	.42839952698747409380e-63	13	7	.1569263805950391e-255

Table 10: For comparison, high precision values of $c_r(k)$, $4 \le k \le 13$, $r \le 7$, computed using the program for method 1 of [CFKRS]. The values here agree with those in Tables 2- 9 computed using our cubic accelerant method with experimentally determined remainders, except they are occasionally slightly off in the last decimal place.

5 Checking the Moment Polynomial Conjectures

Let

$$\mathrm{Data}_k(T) = \int_{T_0}^{\mathrm{T}} |\zeta(1/2+it)|^{2k} dt, \tag{32}$$

with $T_0 = 14.134725\ldots$ being the imaginary part of the first non-trivial zero of zeta, and let

$$\mathrm{Conjecture}_k(T) = \int_{T_0}^{\mathrm{T}} P_k(\log \frac{t}{2\pi}) dt. \tag{33}$$

We used our tables of $c_r(k)$ and a simple integration by parts to compute the latter for given T. Note that because the leading coefficients of $P_k(x)$ are very small, in the range of T considered, the $c_r(k)$'s with midrange and higher values of r contribute the dominant amount to the integral. To accurately compute the prediction of [CFKRS] one does need the lower terms of the polynomial $P_k(x)$. Table 11 displays, for $1 \le k \le 13$ and $T = 10^8$, the r for which the corresponding term in (33) contributes the dominant amount to integral.

k	1	2	3	4	5	6	7	8	9	10	11	12	13
r	0	0	2	6	11	18	26	37	49	64	80	99	119

Table 11: The value of r for each k for which the corresponding term in (33) contributes the dominant amount when $T = 10^8$.

To calculate (32), we used the tanh-sinh quadrature scheme [Ba, BLJ] to accurately estimate each integral between consecutive non-trivial zeros of the zeta function on the critical line. All our computations of (32) were carried out using 64 bit machine doubles. To tabulate all the zeros up to, and slightly beyond, $T = 10^8$, we used Rubinstein's `C++` L-function package lcalc [R]. It applies the Riemann Siegel formula to evaluate $\zeta(1/2+it)$ and look for sign changes of the Hardy Z-function, Brent's method to compute the zeros of zeta [Br], and a variant of Turing's test [E] to confirm that all zeros up to given height have been found.

Figures 1-2 depict the relative remainder term for 1000 values of T between 0 and 10^8, roughly spaced apart every 10^5. More specifically, we let T_j be the imaginary part of the first zero above $10^5 j$, so that $T_j \approx 10^5 j$, and plot the values of

$$\frac{\mathrm{Data}_k(T_j) - \mathrm{Conjecture}_k(T_j)}{\mathrm{Conjecture}_k(T_j)}, \tag{34}$$

for $1 \le j \le 1000$, and $1 \le k \le 12$.

We also calculated the running average of the remainder term divided by the running average of the conjecture:

$$\frac{\sum_{j=1}^{J}(\mathrm{Data}_k(T_j) - \mathrm{Conjecture}_k(T_j))}{\sum_{j=1}^{J}\mathrm{Conjecture}_k(T_j)}. \tag{35}$$

If we define

$$\mathrm{SmoothData}_k(T) = \int_{T_0}^{\mathrm{T}} |\zeta(1/2+it)|^{2k}(1-t/T)dt, \tag{36}$$

and

$$\mathrm{SmoothConjecture}_k(T) = \int_{T_0}^{\mathrm{T}} P_k(\log\frac{t}{2\pi})(1-t/T)dt, \tag{37}$$

then (35) gives a discrete approximation to the smoothed relative remainder:

$$\frac{\mathrm{SmoothData}_k(T_j) - \mathrm{SmoothConjecture}_k(T_j)}{\mathrm{SmoothConjecture}_k(T_j)}. \tag{38}$$

The reason for considering smoothed moments is that the noisy remainder terms of the sharply truncate moments, when averaged, tend to be smaller.

Note that the vertical axes in these figures change from plot to plot to allow us to meaningfully display the relative remainder terms, which as a whole get larger, as k increases. We also set the zoom level to show the running averages in some detail. As a compromise, a few outliers are omitted from these plots for smaller T, roughly up to 10^7, and $k \le 4$.

Table 12 lists the values of $\mathrm{Data}_k(T)$, and $\mathrm{Conjecture}_k(T)$ for $k = 1, \ldots, 13$ and $T = 100000000.64$, the first zero of zeta above 10^8. We also list the values of the averages over all 1000 values of T_j

$$\frac{1}{1000}\sum_{j=1}^{1000}\mathrm{Data}_k(T_j) \tag{39}$$

and

$$\frac{1}{1000}\sum_{j=1}^{1000}\mathrm{Conjecture}_k(T_j). \tag{40}$$

Our data supports the CFKRS conjecture for the full asymptotics of the moments of zeta as described in equation (1), though for larger k, it is difficult to gauge the size of the remainder term.

For $k = 1$ the data suggests an even stronger remainder term of $O\left(T^{1/4+\epsilon}\right)$, supported by the agreement of $\mathrm{Data}_1(T)$ with $\mathrm{Conjecture}_1(T)$ to about 3/4 of the decimal places left of the decimal point for the values of T examined. The relative remainder term is of size around 10^{-6} when $T \approx 10^8$, and, typically, a couple of orders of magnitude smaller when averaged.

For $k = 2$ the data agrees with the conjecture to about half the decimal places, with a relative remainder term of size around 10^{-6}. For $k = 3$, the agreement is to slightly less than half the decimal places.

For fixed T, as k increases, the moments have the effect of amplifying the largest values of $|\zeta(1/2 + it)|$. This can be seen in our plots, for larger k, where the remainder terms are qualitatively the same, with large jumps at the same values of T corresponding to relatively large values of zeta. It therefore becomes more difficult to ascertain, as k grows, whether an upper bound for the remainder term of the form $O_k(T^{1/2+\epsilon})$ holds. Nonetheless, the table and figures reveal an excellent fit for the CFKRS prediction with the moments which persists through to the 24th and 26th moments, where the relative agreement is to within around one to two decimal places.

In some sense, the fit between columns 2 and 3 of Table 12 is better than it ought to be for larger k, for example more than three decimal places for $k = 13$, but only agreeing to around 90% for the average remainder. A quick inspection of the figures reveals that the relative remainder, at $T = 10^8$, happens to, fortuitously, best its neighbours, especially for larger k. Nonetheless, the overall agreement between the CFKRS prediction and our data across all values of k and T, as depicted in the figures, lends strong support to their conjecture for the full asymptotics of the moments of the zeta function.

References

[Ba] David H. Bailey, *Tanh-Sinh High-Precision Quadrature*, Jan 2006; LBNL-60519.

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k	$\text{Data}_k(T)$	$\text{Conjecture}_k(T)$	(39)	(40)
1	1673723690.436	1673723498.495	62463107.03367	62463106.44834
2	637388343407.	637389923500.	22091815715.8	22091815007.3
3	8.04585314342e+14	8.04581403344e+14	2.54969941363e+13	2.54969410053e+13
4	1.7376480696e+18	1.7374512576e+18	5.0233293703e+16	5.0234406051e+16
5	5.0837678819e+21	5.0816645028e+21	1.3436072658e+20	1.3439894636e+20
6	1.815301994e+25	1.813639687e+25	4.400401104e+23	4.406152513e+23
7	7.480512969e+28	7.468884126e+28	1.66796929e+27	1.674460675e+27
8	3.43851173e+32	3.43090327e+32	7.0665341e+30	7.13020297e+30
9	1.72388578e+36	1.71918466e+36	3.26838252e+34	3.32617445e+34
10	9.2785049e+39	9.251733e+39	1.6228859e+38	1.6729912e+38
11	5.2991086e+43	5.2863072e+43	8.5447722e+41	8.967058e+41
12	3.182548e+47	3.179455e+47	4.726347e+45	5.076008e+45
13	1.995625e+51	1.999377e+51	2.726982e+49	3.013406e+49

Table 12: The values of $\text{Data}_k(T)$ and $\text{Conjecture}_k(T)$ at $T = 100000000.64$, the first zero after 10^8. We also display, in the last two columns, the values of the averages over all 1000 values of T_j: $\frac{1}{1000}\sum_{j=1}^{1000} \text{Data}_k(T_j)$, and $\frac{1}{1000}\sum_{j=1}^{1000} \text{Conjecture}_k(T_j)$.

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[R] M. O. Rubinstein, *lcalc - the L-function calculator*, `http://code.google.com/p/l-calc/`.

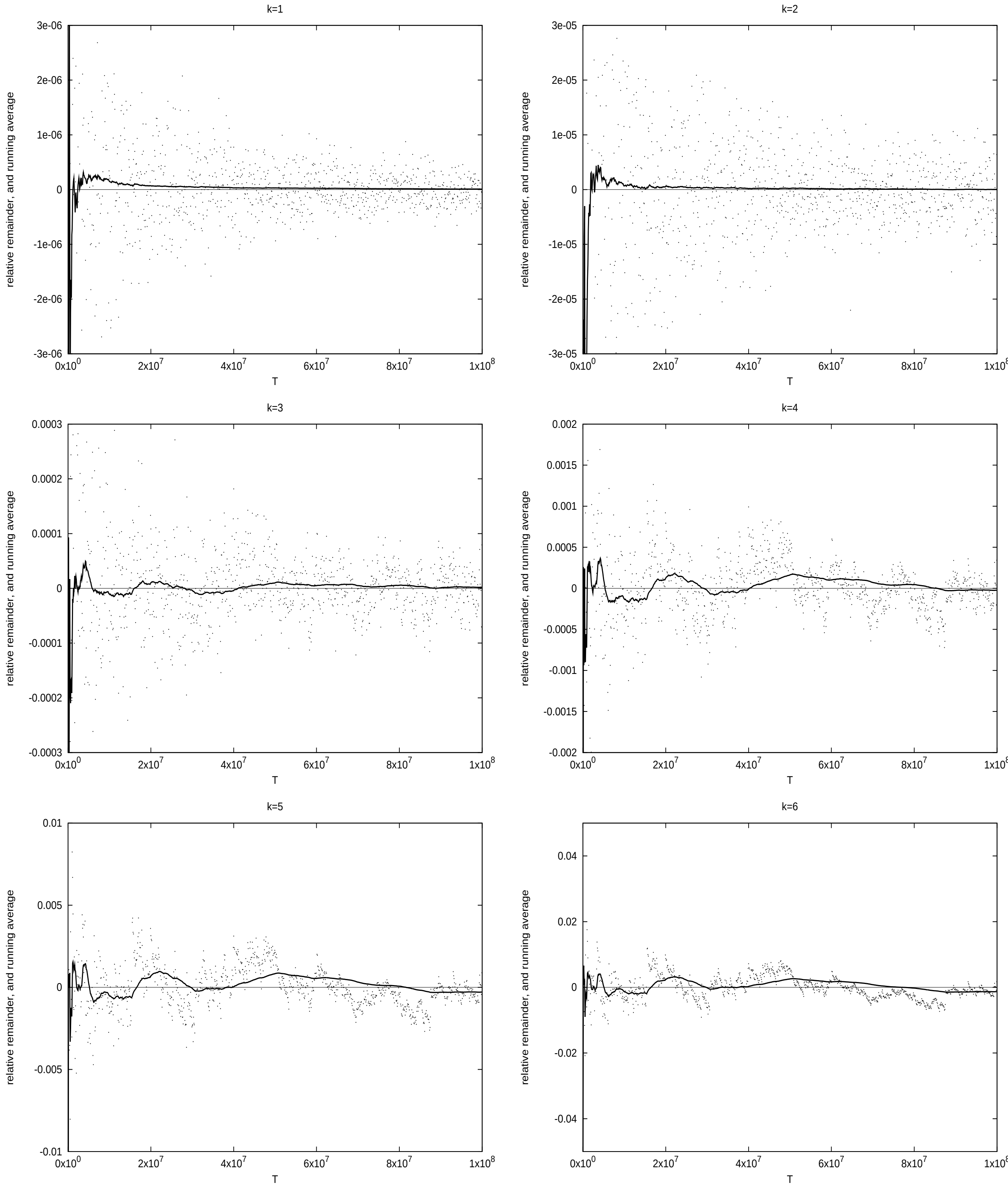

Figure 1: Plot of the relative remainder (34), and running average of the remainder (38), for 1000 values of T, and $1 \leq k \leq 6$. Horizontal axis is T.

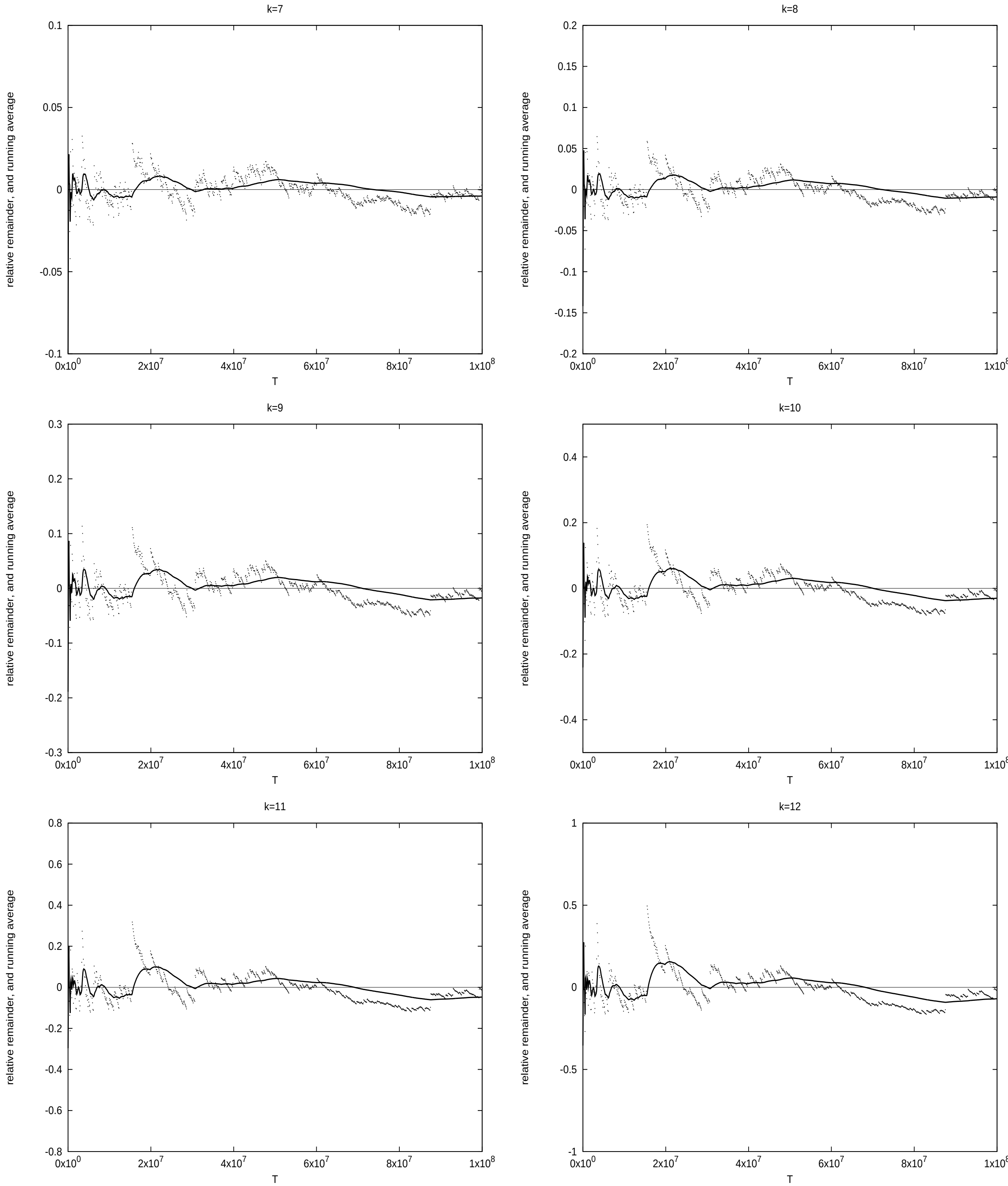

Figure 2: Plot of the relative remainder (34), and running average of the remainder (38), for 1000 values of T, and $7 \leq k \leq 12$. Horizontal axis is T.